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Asymptotic analysis of the lumped-capacitance approximation

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Abstract—The lumped-capacitance (LC) approximation for solid bodies suddenly exposed to constant-property environments is analyzed. A mathematical derivation of the LC approximation is presented for bodies of an arbitrary shape. A Biot number is treated as a small parameter, and asymptotic analyses of the parabolic heat equation are developed in terms of this parameter. The LC approximation is recovered to leading order, while first-order solutions include spatial temperature variations and provide error estimates involved with using the LC approximation. It is also shown that the LC approximation may sometimes be accurate when conventionally-defined Biot numbers are of order unity or greater.

INTRODUCTION

THE SCIENCE of heat transfer has benefitted tremendously from the development of numerical methods. Many problems that are impossible to exactly solve analytically may now be solved numerically—for example, transient heat conduction in irregularly-shaped solid bodies. Though powerful numerical methods are widely available, it is sometimes desirable (e.g. because of convenience or costs) to utilize approximate analytical solutions that introduce small errors but which are nonetheless useful for engineering calculations. An example of this is the lumped-capacitance (LC) approximation used for analysis of transient conduction heat transfer problems involving surface convection effects. The LC approximation assumes that the temperature of a solid body suddenly exposed to a convective environment varies temporally but not spatially; in essence, the solid thermal conductivity is assumed to be infinite. Because of its simplicity, it is desirable to use the LC approximation whenever it may be accurately employed. Hence, it is important to understand the applicability and accuracy of this approximation. To this end, asymptotic analyses are developed in this paper which show how the LC approximation may be formally derived from the parabolic heat equation for bodies of arbitrary shape, and which also provide corrections to the LC approximation. Moreover, it will be shown in separate analyses that under certain circumstances, the LC approximation may be accurately employed when temperature gradients at the surface of a solid are not small.

In the literature, e.g. refs. [1–3], it is generally accepted that temperature gradients in solid bodies being convectively cooled may be neglected if the Biot number (Bi) is sufficiently small. The Biot number is the ratio of the product of the convective heat transfer coefficient h and a characteristic length L to the solid thermal conductivity k (i.e. $Bi = hL/k$), and is typically interpreted as a ratio of solid and convective thermal resistances. The length L is sometimes defined as the ratio of the body volume to the total body surface area. For $Bi \gg 1$, conduction heat-transfer resistance in the solid is argued to be large relative to convective heat-transfer resistance between the solid and the environment, so that temperature gradients in the solid may be 'strong', for $Bi \ll 1$, conduction heat-transfer resistance is argued to be small relative to convective resistance, so that temperature gradients in the solid may be 'gentle'. Applicability of the lumped-capacitance approximation is sometimes justified by arguing that if Bi is sufficiently small, the lumped-capacitance approximation is valid since temperature gradients are gentle and may be neglected. Exact solutions of the governing partial differential equations have been presented for simple geometries [1] and it can be shown that $Bi \ll 1$ yields gentle temperature gradients, supporting the concept of lumped capacitance.

It follows from energy conservation arguments [1] that within the context of the LC approximation, the temperature history of a solid is described by

$$(T - T_\infty)/(T_i - T_\infty) = \exp[-hAt/(\rho Vc)]. \quad (1)$$

In equation (1), t is time, T the instantaneous solid temperature, T_i the initial solid temperature, T_∞ the (constant) environment temperature, A the surface area exposed to the fluid, V the solid volume, ρ the solid density, and c the solid specific heat (all solid properties are assumed constant). In reality, the solid temperature varies in both time and space, and the temperature field may be assumed to be governed by the heat equation

$$\partial T/\partial \tau = \alpha \nabla^2 T \quad (2)$$

where α is the thermal diffusivity (assumed constant).

NOMENCLATURE

A	solid body surface area exposed to the fluid
A_t	total surface area of the solid body
a	inner surface of conical section
Bi	Biot number defined after equations (3) and (28)
b	outer surface of conical section
C_1, C_2	integration constants
c	solid specific heat
f	variable defined after equation (5)
G	variable defined after equation (13)
g	variable defined after equation (5)
$\bar{g}$	variable defined after equation (7)
h	convective heat transfer coefficient
I	variable defined after equation (15)
J_0	Bessel function
j	variable defined after equation (24)
K	variable defined after equation (26)
k	solid thermal conductivity
L	characteristic length in the solid
m	variable defined after equation (29)
$\mathbf{n}$	outward normal unit vector
R_{cond}	conduction thermal resistance
R_{conv}	convective thermal resistance
R_n	variable defined by equation (26)
r	radius
s, s'	dimensionless surface areas defined after equation (3)
T	solid temperature
T_i	initial solid temperature
T_∞	characteristic fluid temperature
t	time
V	solid volume
$\mathbf{x}$	physical coordinate system
$\mathbf{z}$	variable defined after equation (3).

Greek symbols

α	solid thermal diffusivity
β_n	variable defined after equation (21) and in equation (27)
Γ	solid angle of conical section
Θ	dimensionless temperature defined before equation (9)
θ	dimensionless temperature defined before equation (3)
θ_{LC}	dimensionless temperature defined by equation (28)
θ_0	average dimensionless temperature of the solid interior
θ_s	average dimensionless temperature of the surface exposed to the environment
λ_n	variable defined before equation (18)
ρ	density of solid
τ	dimensionless time defined before equation (3) and after equation (24)
τ'	rescaled dimensionless time defined after equation (8)
τ''	rescaled dimensionless time defined after equations (22) and (28)
ϕ	variable defined after equation (24).

To solve equation (2), initial and boundary conditions must be specified. For example, we may impose an initial condition of $T(\mathbf{x}, 0)$, the convective boundary condition $-k\nabla T\cdot\mathbf{n} = h(T-T_\infty)$ along the portion of the surface exposed to the fluid, $\nabla T\cdot\mathbf{n} = 0$ elsewhere along the surface, and that $\nabla T = 0$ somewhere inside the body (e.g. at a symmetry plane or point). Here, $\mathbf{n}$ is an outward normal unit vector at the solid surface, $\mathbf{x}$ denotes position in the material, k is the solid thermal conductivity, and boldface characters denote vector quantities.

In the following section, equation (2) is nondimensionalized and Bi is identified as a controlling parameter. Asymptotic analyses are performed for Bi going to zero. The lumped-capacitance approximation emerges as the lowest-order asymptotic solution, while higher-order solutions display the effects of spatial temperature gradients. In all analyses, material properties are considered constant.

ANALYSIS OF THE HEAT EQUATION

Consider the situation where a solid body is initially at the uniform temperature T_i. Assume that the body is suddenly exposed over some portion of its surface to a convective environment of constant temperature (T_∞) which is characterized by the constant convective heat transfer coefficient h. Where the body surface is not exposed to the environment, it is considered to be insulated. In terms of the dimensionless temperature $\theta = (T-T_\infty)/(T_i-T_\infty)$, the dimensionless time $\tau = \alpha t/L^2$, and the dimensionless spatial coordinates $\mathbf{z} = \mathbf{x}/L$, equation (2) and its boundary and initial conditions are

$$\partial\theta/\partial\tau = \nabla^2\theta, \quad \theta(\mathbf{z}, 0) = 1,$$
$$(\nabla\theta\cdot\mathbf{n}+Bi\,\theta)_s = (\nabla\theta\cdot\mathbf{n})_{s'} = 0 \qquad (3)$$

where $Bi = hL/k$. The spatial coordinates may be properly nondimensionalized by first expressing $\mathbf{x}$ in cartesian coordinates and then forming $\mathbf{z} = \mathbf{x}/L$. Transformation to other coordinate systems (e.g. spherical) may then be performed. In equation (3), the subscript s denotes the nondimensional surface area of the body exposed to the fluid, the subscript s' the remainder of the body surface, and L is selected to be a length which characterizes the conduction path length to the portions of the body furthest-removed from the exposed surface. In terms of physical variables, $s = A/L^2$ and $s' = (A_t - A)/L^2$, where A_t is the total surface area of the body. In problems exhibiting symmetry, such as with spherically symmetric conduction, the boundary condition $\nabla\theta = 0$ may be utilized.

In the following it will be assumed that $Bi \ll 1$, and analyses of equation (3) will be developed for $Bi \to 0$. To proceed, the expansion $\theta = \theta_0 + Bi\,\theta_1 + \cdots$ may be inserted into equation (2). Equating terms of equal order in powers of Bi yields the leading-order problem

$$\partial\theta_0/\partial\tau = \nabla^2\theta_0, \quad \theta_0(\mathbf{z}, 0) = 1,$$
$$(\nabla\theta_0 \cdot \mathbf{n})_s = (\nabla\theta_0 \cdot \mathbf{n})_{s'} = 0. \quad (4)$$

By inspection, the solution to equation (4) is simply $\theta_0 = 1$. The first-order problem is then found to be

$$\partial\theta_1/\partial\tau = \nabla^2\theta_1, \quad \theta_1(\mathbf{z}, 0) = 0,$$
$$(\nabla\theta_1 \cdot \mathbf{n})_s = -1, \quad (\nabla\theta_1 \cdot \mathbf{n})_{s'} = 0. \quad (5)$$

A solution to equation (5) may be written as $\theta_1 = M\tau + f + g$, where M is a constant, and f and g satisfy

$$\nabla^2 f = M, \quad (\nabla f \cdot \mathbf{n})_s = -1, \quad (\nabla f \cdot \mathbf{n})_{s'} = 0 \quad (6)$$

$$\partial g/\partial\tau = \nabla^2 g, \quad (\nabla g \cdot \mathbf{n})_s = (\nabla g \cdot \mathbf{n})_{s'} = 0,$$
$$g(\mathbf{z}, 0) = -f. \quad (7)$$

The variable f is a function of $\mathbf{z}$ only, while g is a function of both $\mathbf{z}$ and τ. Equation (6) determines $f(\mathbf{z})$ only to within an arbitrary constant. Since $f(\mathbf{z})$ and $g(\mathbf{z}, \tau)$ are summed in the equation for θ_1, the value of this constant is immaterial and is taken for simplicity to be zero in the rest of this paper. Applying the divergence theorem to equation (6) yields $M = -s/v$, where $v = V/L^3$ is a nondimensional body volume. Inspection of equation (7) shows that g will approach, for $\tau \to \infty$, the constant solution

$$\bar{g} = (-1/v)\iiint_v f \, dv.$$

The expansion for θ may now be written as

$$\theta = 1 + Bi[-(s/v)\tau + f + g] + \cdots \quad (8)$$

implying that equation (8) is valid for $\tau \ll 1/Bi$. The appearance of $Bi\,\tau$ suggests that the rescaled time $\tau' = Bi\,\tau$ should be introduced for analysis of later periods (s/v is treated as order unity).

In terms of τ' and the nomenclature $\Theta = \theta$, equation (3) becomes

$$Bi\,\partial\Theta/\partial\tau' = \nabla^2\Theta, \quad (\nabla\Theta \cdot \mathbf{n} + Bi\,\Theta)_s = (\nabla\Theta \cdot \mathbf{n})_{s'} = 0. \quad (9)$$

Analysis in the τ' variable is for later periods of time, so the initial condition has been neglected. Equation (8) may be considered an 'inner' solution, while solutions of equation (9) are 'outer' solutions. For analysis, the expansion $\Theta = \Theta_0 + Bi\,\Theta_1 + Bi^2\,\Theta_2 + \cdots$ may be inserted into equation (9), yielding the leading-order problem

$$\nabla^2\Theta_0 = 0, \quad (\nabla\Theta_0 \cdot \mathbf{n})_s = (\nabla\Theta_0 \cdot \mathbf{n})_{s'} = 0. \quad (10)$$

The solution to equation (10) is $\Theta_0 = \Theta_0(\tau')$ such that Θ_0 is an undetermined function of τ'. The first-order problem is then

$$d\Theta_0/d\tau' = \nabla^2\Theta_1, \quad (\nabla\Theta_1 \cdot \mathbf{n} + \Theta_0)_s = (\nabla\Theta_1 \cdot \mathbf{n})_{s'} = 0. \quad (11)$$

Equation (11) may be integrated over the body volume, yielding (after application of the divergence theorem)

$$d\Theta_0/d\tau' + (s/v)\Theta_0 = 0. \quad (12)$$

Equation (12) is the lumped-capacitance approximation. Its solution is $\Theta_0 = C_1 \exp(-s\tau'/v)$, where C_1 is a constant that will be determined by matching to the inner solution.

The solution to equation (11) may be written as

$$\Theta_1 = fC_1 \exp(-s\tau'/v) + G. \quad (13)$$

In equation (13), G is a function of τ' only and may be evaluated by considering the second-order problem

$$-(s/v)fC_1 \exp(-s\tau'/v) + (dG/d\tau') = \nabla^2\Theta_2,$$
$$(\nabla\Theta_2 \cdot \mathbf{n} + \Theta_1)_s = (\nabla\Theta_2 \cdot \mathbf{n})_{s'} = 0 \quad (14)$$

and f is defined after equation (5). Equation (14) may be integrated over the solid volume to yield

$$(dG/d\tau') + (s/v)G = C_1 I \exp(-s\tau'/v) \quad (15)$$

where

$$I = (s/v^2)\iiint_v f \, dv - (1/v)\iint_s f \, ds.$$

The solution to equation (15) may be written as $G = IC_1(vC_2/s + \tau')\exp(-s\tau'/v)$, where C_2 is a constant of integration. The outer expansion may therefore be written as

$$\Theta = C_1 \exp(-s\tau'/v)[1 + Bi(f + IvC_2/s + I\tau')] + \cdots. \quad (16)$$

Matching equations (8) and (16) yields $C_1 = 1$ and $C_2 = s\bar{g}/(Iv)$. A composite solution (which satisfies the initial condition $\theta(\mathbf{z}, 0) = 1$) may then be written as

$$\theta = \exp(-Bi\,\tau s/v)[1 + Bi(f + \bar{g} + I\,Bi\,\tau)] + Bi(g - \bar{g}) + \cdots. \quad (17)$$

In equation (17) the lumped-capacitance solution is simply the leading-order term, $\exp(-Bi\,\tau s/v)$, and first-order corrections are of the order of Bi. The appearance of $Bi^2\,\tau$ suggests that equation (17) is valid for $\tau \ll 1/Bi^2$. For description of temperatures at later times, more analysis may be needed, as described below.

APPLICATIONS

We may use equation (17) to analyze transient spherically-symmetrical conduction in a solid sphere whose entire surface is suddenly exposed to a convective environment. For this case we may select $L = V/A_t$ so that $s/v = 1$. The governing equation for f is then

$$d^2f/dz^2+(2/z)\,df/dz = -1, \quad (df/dz)_{z=3} = -1$$

where z is a nondimensional radial coordinate ($0 \leqslant z \leqslant 3$). The solution for f is $-z^2/6$. Using this, g may be found with standard mathematical techniques [4] to be

$$g = 9/10+1/(81z)\sum_{n=1}^{\infty}\exp(-\lambda_n^2\tau)(9+1/\lambda_n^2)$$
$$\times\sin(\lambda_n z)\int_0^3 z^3\sin(\lambda_n z)\,dz$$

where λ_n values are roots of $3\lambda_n = \tan 3\lambda_n$, and $\lambda_1 > 0$. The integral I is found to have the value 3/5. The temperature field in the sphere is then approximated by

$$\theta = \exp(-Bi\,\tau)[1+Bi(9/10-z^2/6+3Bi\,\tau/5)]$$
$$+Bi(g-9/10)+\cdots. \quad (18)$$

Equation (18) illustrates that temperature gradients in the solids relax to quasisteady profiles over timescales of $\tau \leqslant O(1)$ while bulk temperature changes occur over timescales of $\tau = O(1/Bi)$.

Equation (18) may also be derived by expanding the exact solution [1] for $Bi \to 0$. When this is done, it is found that the terms $1+3Bi^2\,\tau/5$ appearing in equation (18) arise as an expansion of $\exp(3Bi^2\,\tau/5)$, which occurs naturally as a multiplier of $\exp(-Bi\,\tau)$ in the expansion of the exact solution. In equation (17), $1+I\,Bi^2\,\tau$ may then be an expansion of a term such as $\exp(I\,Bi^2\,\tau)$, which would multiply $\exp(-Bi\,s\tau/v)$. It may be possible to formally recover a term analogous to $\exp(I\,Bi^2\,\tau)$ with multiple-scale theory (see, e.g. ref. [5]), which should render the expansion uniformly valid for all times. Such analyses have not been performed. It is interesting to note, though, that expanding $\exp(I\,Bi^2\,\tau)[1+Bi(f+\bar{g})]$ for $Bi \to 0$ yields $1+Bi(f+\bar{g})+I\,Bi^2\,\tau+O(Bi^3)$. If we use this result in equation (17), the following expression is obtained:

$$\theta = \exp[-Bi\,\tau s/v(1-I\,Bi\,v/s)][1+Bi(f+\bar{g})]$$
$$+Bi(g-\bar{g})+\cdots. \quad (19)$$

Substitution shows that equation (19) satisfies the original partial differential equation and initial and boundary conditions (equation (3)) to first order, and is thus a valid asymptotic solution. Equation (19) appears in addition to be uniformly valid in τ.

In reference to equation (19), equation (18) may be rewritten as

$$\theta = \exp[-Bi\,\tau(1-3\,Bi/5)][1+Bi(9/10-z^2/6)]$$
$$+Bi(g-9/10)+\cdots. \quad (20)$$

For an infinitely long cylinder whose entire surface is suddenly exposed to the convective environment, the corresponding solution (based on equation (19)) is found to be

$$\theta = \exp[-Bi\,\tau(1-Bi/2)][1+Bi(1/2-z^2/4)]$$
$$+(Bi/18)\sum_{n=1}^{\infty}\exp(-\beta_n^2\tau)[J_0(\beta_n z)/J_0(2\beta_n)]$$
$$\times\int_0^2 z^3 J_0(\beta_n z)+\cdots \quad (21)$$

where z is the nondimensional radial coordinate ($0 \leqslant z \leqslant 2$), J_0 is a Bessel function, and $dJ_0/dz = 0$ at $z = 2\beta_n$. For an infinite slab suddenly exposed on both sides to a convective environment, it is found that equation (19) yields

$$\theta = \exp[-Bi\,\tau(1-Bi/3)][1+Bi(1/6-z^2/2)]$$
$$+2\,Bi\sum_{n=1}^{\infty}[(-1)^n/(n\pi)^2]\cos(n\pi z)\exp(-n^2\pi^2\tau)+\cdots \quad (22)$$

where z is the nondimensional planar coordinate as measured from the slab midplane ($-1 \leqslant z \leqslant 1$). In equations (21) and (22), $L = V/A_t$ such that $s/v = 1$. Equations (21) and (22) may also be derived by expanding the exact solutions [1] for $Bi \to 0$.

Shown in Fig. 1 are comparisons of equation (22)

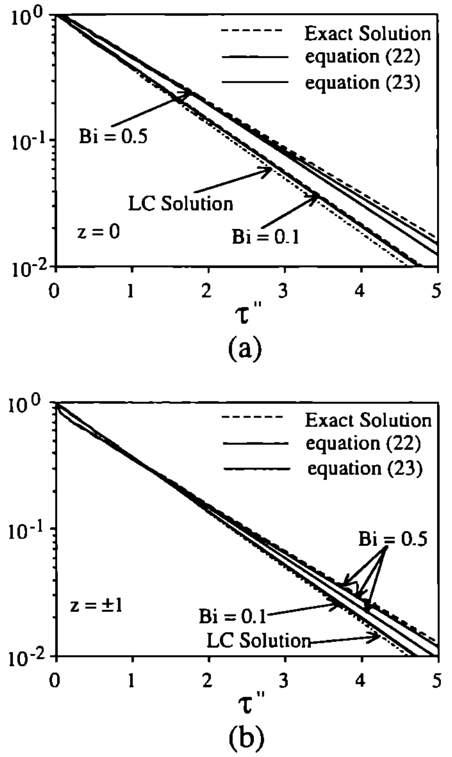

FIG. 1. Comparison of equations (22) and (23) and the LC and exact solutions for transient conduction in a plane wall at: (a) $z = 0$; (b) $z = \pm 1$.

and the exact solution [1] for conduction in a plane wall. For $\tau \geqslant 1$, the infinite sum in equation (22) is negligible relative to the first term. The LC solution (equation (1)) is also plotted for comparison, as is the asymptotic solution based on equation (17), which is given by

$$\theta = \exp(-Bi\,\tau)[1+Bi(1/6-z^2/2+Bi\,\tau/3)] + 2\,Bi \sum_{n=1}^{\infty} [(-1)^n/(n\pi)^2]\cos(n\pi z) \times \exp(-n^2\pi^2\tau)+\cdots. \quad (23)$$

The timescale is selected to be $\tau'' = Bi\,\tau$ such that the right-hand side of equation (1) is $\theta = \exp(\tau'')$. For $Bi = 0.1$, differences between the exact solution and equations (22) and (23) are difficult to distinguish on the graph. In Fig. 1 it is evident that equation (22) provides good results even for Bi as large as 0.5. Results from equation (23) are an improvement over the LC solution, but are not as accurate as equation (22).

ANALYSIS FOR $Bi \geqslant O(1)$

In this section, it will be demonstrated that circumstances may exist where the LC approximation is accurate even if Bi as previously defined is of order unity or greater. To illustrate concepts, a specific problem will be posed and solved. The problem to be considered is for spherically-symmetrical conduction in a conical section of a hollow sphere. The inner surface (at the radius $r = a$) is exposed to a fluid characterized by the constant temperature T_∞ and the constant heat transfer coefficient h. The outer surface (at $r = b$) is thermally insulated. The heat equation and boundary and initial conditions for the solid may be expressed in nondimensional form as

$$\partial(\theta z)/\partial\tau = \partial^2(\theta z)/\partial z^2, \quad \theta(z,0) = 1,$$
$$(\partial\theta/\partial z-\phi\theta)_{z=j} = (\partial\theta/\partial z)_{z=1} = 0 \quad (24)$$

where for this analysis $z = r/b$, $j = a/b$, $\phi = hb/k$, $\tau = kt/(\rho c b^2)$, and $\theta = (T-T_\infty)/(T_i-T_\infty)$. Equation (24) has the exact solution [4]

$$\theta = (2/z)\sum_{n=1}^{\infty} \exp(-\beta_n^2\tau)R_n(z)\int_j^1 zR_n(z)\,\mathrm{d}z \quad (25)$$

where

$$R_n(z) = \frac{[1+\beta_n^2]^{1/2}[K\sin(z-j)\beta_n+j\beta_n\cos(z-j)\beta_n]}{[(1-j)(j^2\beta_n^2+K^2)(\beta_n^2+1)+(K-j)(j\beta_n^2-K)]^{1/2}} \quad (26)$$

$K = j\phi+1$, and $\pm\beta_n$, $n = 1, 2, \ldots$ are roots of

$$\tan[(1-j)\beta_n]-\beta_n(K-j)/(j\beta_n^2+K) = 0. \quad (27)$$

In terms of the nondimensional variables, the LC solution (equation (1)) may be rewritten as

$$\theta_{LC} = \exp(-\tau'') \quad (28)$$

where $\tau'' = 3\phi j^2\tau/(1-j^3)$ is a nondimensional LC timescale. We may also define a Biot number as $Bi = hL/k = \phi(1-j)$, where $L = b-a$.

Plotted in Figs. 2(a) and (b) are θ_{LC} and equation (25) for $\phi = 10$ and various values of j. The series in equation (25) was summed until remaining terms were negligible. The results in Figs. 2(a) and (b) are for the inner surface ($z = j$). Figures 3(a) and (b) compare the exact and LC solutions for $\phi = 10$ and various values of j, but at $z = 1$ (the outer surface). In Figs. 2 and 3, it is evident that the LC approximation yields accurate results for $j \to 0$ or 1. For intermediate values of j, errors are significant. Though not presented, other calculations showed that as ϕ decreases, errors decrease. For $\phi \leqslant 0.1$, differences between the LC and exact solutions were very small for all values of j. It is stressed that the increased accuracy of the LC solution for $j \to 0$ did not require $Bi \ll 1$. In fact, for these results $Bi \to 10$ as $j \to 0$.

These trends may be understood if the problem is reanalyzed in terms of thermal resistances. To begin, we may note that equation (1) defines $\rho Vc/(hA)$ as a characteristic time for significant lumped temperature changes to occur. If this time is large relative to the characteristic time L^2/α for thermal waves to propagate through the body (i.e. $hAL^2/(kV) \ll 1$) it may be concluded that nearly quasisteady temperature profiles are set up in the solid for $\tau'' \ll 1$. These profiles

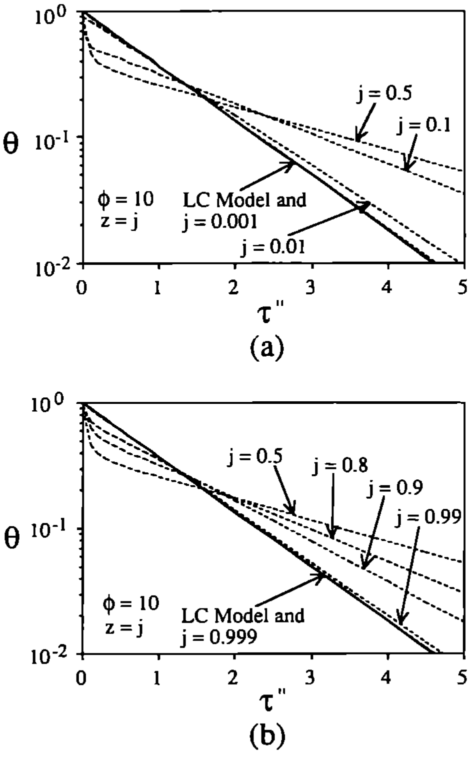

FIG. 2. Comparison of the LC (solid lines) and exact (dashed lines) solutions at $z = j$ for: (a) $0.001 \leqslant j \leqslant 0.5$; (b) $0.5 \leqslant j \leqslant 0.999$.

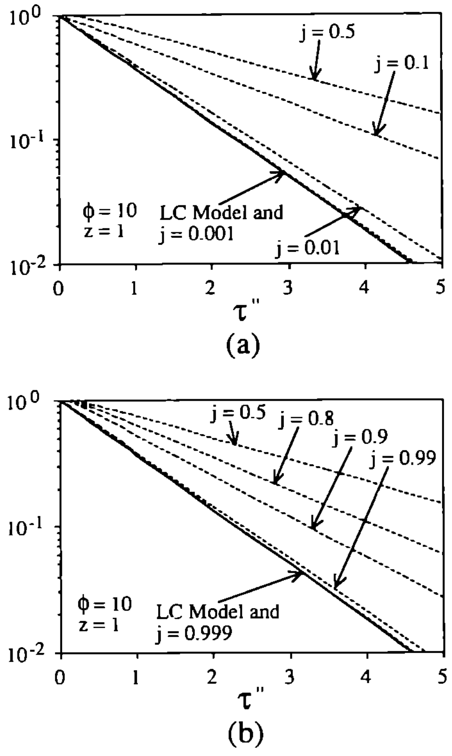

FIG. 3. Comparison of the LC (solid lines) and exact (dashed lines) solutions at $z = 1$ for: (a) $0.001 \leqslant j \leqslant 0.5$; (b) $0.5 \leqslant j \leqslant 0.999$.

will exist over most of the cooling (or heating) history of the solid. For further analysis, assume that the bulk of the interior is at the average nondimensional temperature θ_o, while the exposed surface is at the nondimensional temperature θ_s. Assume that the thermal resistance R_{cond} for conduction heat transfer from the interior to the surface exposed to the environment may be defined, as well as the convective heat transfer resistance R_{conv}. In terms of these thermal resistances, it then follows that $\theta_o - \theta_s = \theta_o m/(1+m)$, where $m = R_{cond}/R_{conv}$ is a Biot number. From energy conservation, the time rate of change of θ_o may be assumed to be proportional to θ_s such that the following equation will approximately hold:

$$d\theta_o/d\tau'' + \theta_o/(1+m) = 0. \tag{29}$$

The solution to equation (29) is

$$\theta_o = \exp[-\tau''/(1+m)] \tag{30}$$

where the initial condition $\theta_o(\tau'' = 0) = 1$ has been applied. Equation (30) may be viewed as a corrected version of the LC solution; it approximately accounts for the finite resistance to conduction heat transfer out of the solid. If $m \ll 1$, corrections are negligible for $\tau'' \ll 1/m$. For θ_s, we may write

$$\theta_s \approx \theta_o/(1+m). \tag{31}$$

Use of equation (31) for accurately describing temperatures for $\tau'' \geqslant O(1)$ requires that quasisteady temperature profiles near the solid surface exposed to the convective environment be established for times $\tau'' \ll 1$.

For the problem described by equation (24), it will be assumed that $R_{cond} = (T_o - T_s)/[k\Gamma a^2(\partial T/\partial r)_{r=a}]$ and $R_{conv} = 1/(k\Gamma a^2)$, where Γ is the solid angle of the conical section and temperatures are evaluated by considering only the first term in equation (25). It then follows that

$$m = [(1+j\phi)/\beta_1] \sin[(1-j)\beta_1] + j\cos[(1-j)\beta_1] - 1.$$

Equations (25), (28), (30) and (31) are compared in Figs. 4 and 5 for $\phi = 10$ and various values of j. In these figures it is evident that equations (30) and (31) can provide reasonable corrected expressions for predicting nondimensional temperatures (differences between equations (25) and (30) or (31) for $j = 0.1$ and 0.99 are difficult to distinguish on the graph). There are two values of j which produce a given value of m (see Fig. 6). Interestingly, it is the smaller of the two j values for which equations (30) and (31) produce better results. This is probably because the expression used for m is more accurate for $j \to 0$ than for $j \to 1$. If more accurate expressions for m were used for j near unity, it is expected that the corrected expressions would be more accurate. It is noted that equations (30) and (31) provide quite good results for j as large as 0.2, which corresponds to $m = 1.44$. For $j = 0.5$ (and $m = 1.90$) equations (30) and (31) provide acceptable order-of-magnitude results for θ.

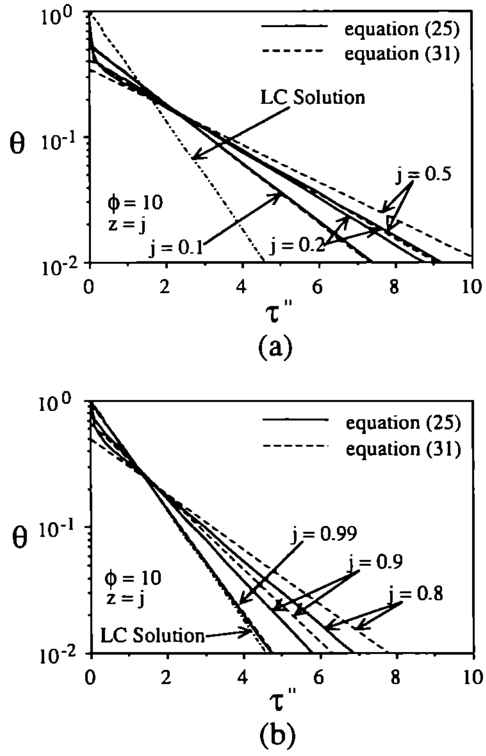

FIG. 4. Comparison of LC, exact, and corrected LC solutions at $z = j$ for: (a) $0.001 \leqslant j \leqslant 0.5$; (b) $0.8 \leqslant j \leqslant 0.99$.

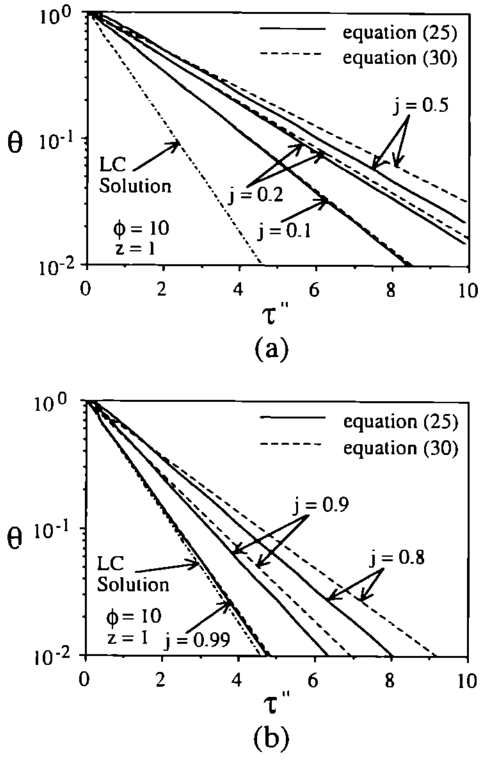

FIG. 5. Comparison of LC, exact, and corrected LC solutions at $z = 1$ for: (a) $0.001 \leqslant j \leqslant 0.5$; (b) $0.8 \leqslant j \leqslant 0.99$.

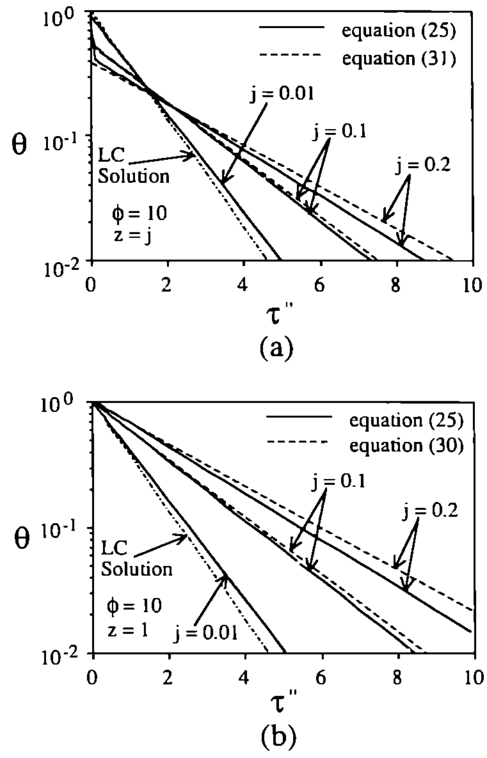

FIG. 7. Comparison of LC, exact and corrected LC solutions with $m = \phi j(1-j)$.

It is expected in general that if $hAL^2/(kV) \ll 1$ and if m is not too large and can be well estimated, the correction formulas (equations (30) and (31)) should be applicable to other problems that cannot be exactly solved analytically. For example, we may suppose that equation (25) cannot be solved exactly, making it difficult to evaluate m. For $j \ll 1$, though, it should be reasonable to assume that the quasisteady temperature profiles are similar to steady-state spherically-symmetric conduction profiles such that $R_{\text{cond}} \approx (1/a - 1/b)/(\Gamma k)$, yielding $m = \phi j(1-j)$. As shown in Fig. 7, use of this relation produces very good estimates for transient temperatures for $j \leqslant 0.1$ ($m \leqslant 0.9$) (differences between equations (25) and (30) or (31) for $j = 0.01$ are difficult to distinguish on the graph). For larger values of j, agreement with equation (25) is typically within an order of magnitude.

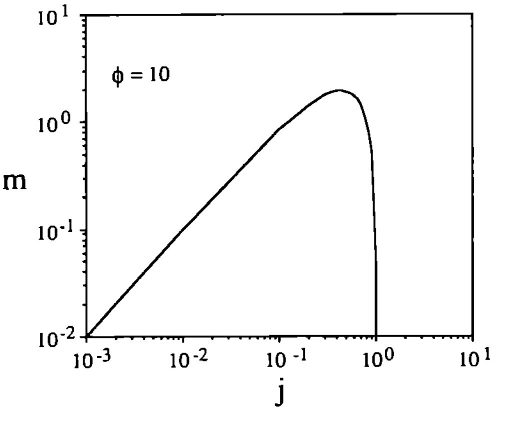

FIG. 6. Plot of m vs j.

It is interesting to note that the LC approximation may be recovered in an asymptotic sense by expanding equations (25)–(28) for the limit $j \to 0$. Expansion of equation (27) yields the result that for the first root, $\beta_1^2 \approx 3j^2\phi \ll 1$, while for the second root, $\pi^2 < (1-j)^2\beta_2^2 < 9\pi^2/4$. Therefore, terms in the series for $n > 1$ are negligible for τ of order unity or greater. This is reasonable, since (for $j \ll 1$) τ of order unity defines characteristic times for propagation of thermal waves through the body. For $j \to 0$, R_1 is approximately $z(3/2)^{1/2}$, and the first integral in equation (25) has the approximate value $(1/6)^{1/2}$. Substitution of these relations into equation (25) yields $\theta \approx \exp(-3j^2\phi\tau)$. This relation, which exhibits no spatial dependence, is the leading-order term in an expansion of equation (28) for $j \to 0$. Further analysis shows that θ differences between the inner surface and the bulk of the body are of order ϕj. It is also found that nondimensional temperature gradients are of order ϕ in a thin region near the inner surface (where z is of order j). Outside this region, nondimensional temperature gradients are small relative to unity.

SUMMARY AND CONCLUSIONS

Transient heat transfer in an isotropic body suddenly exposed to a convective and constant property environment has been studied. The heat equation was nondimensionalized, and asymptotic analyses were performed for $Bi \to 0$ for bodies of arbitrary shape. The leading-order solution recovered the lumped-capacitance (LC) approximation. The first-order solution included the effects of non-zero temperature gradients in the body, and provided error estimates for the LC approximation.

The exact solution for the temperature history of a conical section of a hollow sphere exposed on its interior surface to a convective environment was analyzed. Results showed that the LC solution may be recovered from the exact solution even if non-dimensional temperature gradients at the solid surface are not small relative to unity. If Bi is properly formulated as a ratio of conductive and convective thermal resistances, applicability of the LC approximation is easily deduced.

An approximate theory was developed to provide corrections to the LC approximation. When $R_{\text{cond}}/R_{\text{conv}}$ is well estimated and is not too large, the approximate theory provides good results. For the situation considered here, $R_{\text{cond}}/R_{\text{conv}} < 1$ provided accurate results.

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